Morley's Theorem[1]

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CHAPTER 1

Introduction

The aim of this set of notes is to give a proof of Morley's Theorem:

THEOREM 1.1. *If a complete theory T in a countable language is categorical in one uncountable cardinal, then it is categorical in all uncountable cardinals.*

This is quite a striking result; but perhaps more important than the result itself is the whole range of ideas and techniques that are needed to prove it. In fact, these ideas form the starting point of stability theory and hence of "modern" model theory.

The basic steps of the proof are:

(1) We first show that complete theories in a countable language that are κ-categorical for some uncountable cardinal κ are *totally transcendental*. The proof of this fact relies on the notion of *(order) indiscernible*, a concept which will also play an important role at other points in the proof.
(2) Having established that the theories we are interested in are totally transcendental, we embark on an analysis of these totally transcendental theories. We essentially need two results about these theories: one on atomic extensions, which is relatively easy to establish, while the other on the existence of indiscernibles requires us to develop the rather technical notions of *Morley rank* and *Morley degree*.
(3) After we have done this, we can bring everything together and establish Morley's Theorem.

Some conventions:

- *All theories T will be assumed to be complete and to have infinite models.* (Such theories have no finite models; why?)
- By a *countable theory* we will mean a theory formulated in a countable language.

CHAPTER 2

Indiscernibles

1. Models realizing few types

As a first step towards proving Morley's Theorem we want to establish the following result:

THEOREM 2.1. *Let T be a countable theory and let κ be an infinite cardinal. Then T has a model of cardinality κ which realises only countably many types over every countable subset.*

The proof will take up the whole chapter. A key notion that we need is that of an *indiscernible*:

DEFINITION 2.2. Let I be a linear order and A be an L-structure. A family of elements $(a_i)_{i\in I}$ (or tuples of elements, all of the same length) is called a *sequence of indiscernibles* if for all formulas $\varphi(x_1,\ldots,x_n)$ and all $i_1<\ldots<i_n$ and $j_1<\ldots<j_n$ from I we have

$$A\models\varphi(a_{i_1},\ldots,a_{i_n})\leftrightarrow\varphi(a_{j_1},\ldots,a_{j_n}).$$

Another way of putting this is: the truth of $\varphi(a_{i_1},\ldots,a_{i_n})$ in A depends solely on the order of $i_1,\ldots,i_n$ in I.

REMARK 2.3. Strictly speaking we have defined the notion of an *order indiscernible* in Definition 2.2; the term indiscernible is often used for something else. But as this other notion plays no role in this lecture we have felt free to write indiscernible for what should perhaps be more correctly called order indiscernible.

DEFINITION 2.4. Let I be an infinite linear order and $\mathcal{I}=(a_i)_{i\in I}$ be a sequence of elements in M, $A\subseteq M$. The *Ehrenfeucht-Mostowski type* $\mathrm{EM}(\mathcal{I}/A)$ of $\mathcal{I}$ over A is the set of L-formulas $\varphi(x_1,\ldots,x_n)$ with $M\models\varphi(a_{i_1},\ldots,a_{i_n})$ for all $i_1<\ldots<i_n$.

Note that if $(a_i)_{i\in I}$ is a sequence of indiscernibles, then the Ehrenfeucht-Mostowski type $\mathrm{EM}(\mathcal{I}/A)$ is a complete type.

The proof of Theorem 2.1 depends on two lemmas, the first of which is an important result in its own right and depends on Ramsey's Theorem from combinatorics (see Appendix A).

LEMMA 2.5. (The Standard Lemma) *Let I and J be two infinite linear orders and $\mathcal{I}=(a_i)_{i\in I}$ be a sequence of distinct elements of an L-structure M. Then there is a structure $N\equiv M$ with an indiscernible sequence $(c_j)_{j\in J}$ realizing the Ehrenfeucht-Mostowski type* $\mathrm{EM}(\mathcal{I}/A)$.

PROOF. Choose a set $C=(c_j)_{j\in J}$ of new constants; note that C inherits the structure of a linear order from J. We need to show that

$$\mathrm{Th}(M)\cup\{\varphi(\overline{c})\colon\varphi(\overline{x})\in\mathrm{EM}(\mathcal{I}/A)\}\cup\{\varphi(\overline{c})\leftrightarrow\varphi(\overline{d})\colon\overline{c},\overline{d}\in C\}$$

is consistent. (Here the $\varphi(\overline{x})$ are L-formulas and $\overline{c},\overline{d}$ tuples of elements from C in increasing order.)

By compactness it is sufficient to show that

$$\mathrm{Th}(M) \cup \{\varphi(\overline{c}) : \varphi(\overline{x}) \in \mathrm{EM}(\mathcal{I}/A), \overline{c} \in C_0\} \cup \\ \{\varphi(\overline{c}) \leftrightarrow \varphi(\overline{d}) : \varphi(\overline{x}) \in \Delta, \overline{c}, \overline{d} \in C_0\}$$

has a model, where C_0 and Δ are finite. In addition, we may assume that all tuples $\overline{c}$ have the same length n.

Write $[A]^n$ for the set of n-element subsets of A. Since A is linearly ordered, we can also regard this as the set of n-tuples from A in increasing order. Define an equivalence relation $\sim$ on $[A]^n$ by

$$\overline{a} \sim \overline{b} \Leftrightarrow M \models \varphi(\overline{a}) \leftrightarrow \varphi(\overline{b}) \text{ for all } \varphi(x_1, \dots, x_n) \in \Delta$$

where $\overline{a}, \overline{b}$ are tuples in increasing order. Since this equivalence relation has at most $2^{|\Delta|}$ equivalence classes, Ramsey's Theorem implies that there is an infinite subset B of A with all n-elements subsets in the same equivalence class. Interpret $c \in C_0$ by elements b_c in B ordered in the same way as the c. Then $(M, b_c)_{c \in C_0}$ is a model. □

Corollary 2.6. *Let T be a theory and I be a linear order. Then T has a model with a sequence $(a_i)_{i \in I}$ of distinct indiscernibles.*

Definition 2.7. Let M be an L-structure and $A \subseteq M$ be a subset. Then M is *generated* by A if every element in M is the denotation of some term with parameters from A.

Lemma 2.8. *Assume L is countable. If the L-structure M is generated by a well-ordered sequence $(a_i)_{i \in I}$ of indiscernibles, then M realises only countably many types over every countable subset of M.*

Proof. Let B be a countable subset of M. Then every element in B is the denotation of a term with parameters from $A = \{a_i : i \in I\}$. In fact, because B is countable, there is a countable subset $A_0 \subseteq A$, say $A_0 = \{a_i : i \in I_0\}$ for some countable set $I_0 \subseteq I$, such that every element in B is the denotation of some term with parameters from A_0. This means that the type $\mathrm{tp}(m/B)$ for some $m \in M$ is completely determined by the type $\mathrm{tp}(m/A_0)$. Now, every $m \in M$ is of the form $t\overline{a}$ for some term t and some $a_{i_1}, \dots, a_{i_n}$ in A. This means that the type $\mathrm{tp}(m/A_0)$ is completely determined by the term t and $\mathrm{tp}(a_{i_1}, \dots, a_{i_n}/A_0)$. For the term t there are only countably many possibilities, as the language is countable, and the n-type $\mathrm{tp}(a_{i_1}, \dots, a_{i_n}/A_0)$ is completely determined by:

- the relative position of the i_k;
- the position of i_k relative to I_0, for which there are the following possibilities: (i) bigger than all elements in I_0, (ii) equal to some element in I_0, (iii) smaller than some $i_0 \in I_0$, but bigger than all $\{i \in I_0 : i < i_0\}$.

All in all, this means that there are only countably many possibilities for $\mathrm{tp}(a_1, \dots, a_n/A_0)$ and hence also only countably many possibilities for $\mathrm{tp}(m/A_0)$ and $\mathrm{tp}(m/B)$. □

We are now ready to prove the desired result:

Theorem 2.9. *Let T be a countable theory and let κ be an infinite cardinal. Then T has a model of cardinality κ which realises only countably many types over every countable subset.*

Proof. Let T' be the skolemisation of T in a richer language $L' \supseteq L$, and let I be a well-ordering of cardinality κ and N' be a model of T' with indiscernibles $(a_i)_{i \in I}$. Then the Skolem hull M' generated by $(a_i)_{i \in I}$ has cardinality κ and is an elementary substructure of N'.

In addition, it realises only countably many types over every countable subset by the previous lemma. But then the same is certainly also true for the reduct $M = M' \upharpoonright L$. $\square$

2. Exercises

Exercise 1. Show that a sequence of elements in $(\mathbb{Q}, <)$ is indiscernible if and only if it is constant, strictly increasing or strictly decreasing.

Exercise 2. Let I be an infinite linear order and $\kappa = |I|$. Show that if M is κ-saturated, then there is is a sequence of indiscernibles $(a_i : i \in I)$ in M. (This is Marker 5.5.4.)

Hint: You may want to first prove the following: suppose M is κ-saturated and $(y_i : i \in I)$ is a collection of variables with $|I| \leq \kappa$. If Γ is a collection of formulas all whose free variables belong to $(y_i : i \in I)$ and each finite subset of Γ is realised in M, then Γ is realised in M.

CHAPTER 3

Uncountable categoricity

1. Stable and totally transcendental theories

One of the key steps in proving Morley's Theorem is the proof that countable theories that are categorical in some uncountable cardinal are both ω-stable and totally transcendental. In fact, as we shall see this is an immediate consequence of the main result of the previous chapter. Before we can show this, however, we first need to define ω-stable and totally transcendental theories and show that they are closely related.

Definition 3.1. Write $\{0,1\}^*$ for the set of finite sequences consisting of zeros and ones, and $s0$ and $s1$ for the result of adding 0 and 1 to the end of the sequence $s \in \{0,1\}^*$. A theory T is *totally transcendental* if there is no model M of T with a collection $\{\varphi_s(\overline{x}) \colon s \in \{0,1\}^*\}$ of L_M-formulas such that for any $s \in \{0,1\}^*$:

(1) $M \models \exists \overline{x}\, \varphi_s(\overline{x})$.
(2) $M \models \big(\varphi_{s0}(\overline{x}) \vee \varphi_{s1}(\overline{x})) \to \varphi_s(\overline{x})\big)$.
(3) $M \models \neg\big(\varphi_{s0}(\overline{x}) \wedge \varphi_{s1}(\overline{x})\big)$.

Definition 3.2. Let κ be an infinite cardinal. A theory T is *κ-stable* if in each model M of T and for each $n \in \mathbb{N}$ there are at most κ many n-types over each set of parameters of size at most κ from M.

Theorem 3.3. *A countable theory is totally transcendental if and only if it is ω-stable. In fact, we have:*

(1) *If a theory is ω-stable, then it is totally transcendental.*
(2) *If an L-theory T is totally transcendental and $\kappa \geq |L|$, then T is κ-stable.*

Proof. (1) In a binary tree of L_M-formulas only countably many parameters from M occur; but its existence implies that there are at least 2^ω different types over this countable set.

(2) Let $\kappa \geq |L|$ and assume that T is an L-theory that is not κ-stable. Then there is a model M of T, a set of parameters A in M with $|A| \leq \kappa$ such that there are more than κ many types over A. Write $T_A = \mathrm{Th}_A(M)$ and consider the Stone space $S_n(T_A)$. We will call a formula $\varphi(\overline{x}) \in L_A$ *big* if $|[\varphi]| > \kappa$. Since $|S_n(T_A)| > \kappa$ by assumption and there are κ-many L_A-formules, there must be at least one big formula. The proof will be finished once we show that for any big formula $\varphi(\overline{x}) \in L_A$ there is a formula $\psi(\overline{x}) \in L_A$ such that both $\varphi \wedge \psi$ and $\varphi \wedge \neg\psi$ are big. For then we can use this to create a binary tree of formulas showing that T is not totally transcendental.

So assume that $\varphi \in L_A$ is big, but there is no formula $\psi \in L_A$ such that both $\varphi \wedge \psi$ and $\varphi \wedge \neg\psi$ are big. But since at least one of the two is big, this means that for any $\psi \in L_A$ precisely one of $\varphi \wedge \psi$ and $\varphi \wedge \neg\psi$ is big. Define:

$$p(\overline{x}) = \{\, \psi(\overline{x}) : \varphi(\overline{x}) \wedge \psi(\overline{x}) \text{ is big} \,\}.$$

This defines a complete type and if $\psi \notin p$, then $|[\varphi \wedge \psi]| \leq \kappa$. But then

$$[\varphi] = \bigcup_{\psi \notin p} [\varphi \wedge \psi] \cup \{p\}$$

is the union of at most κ many sets of size at most κ and hence of a size at most κ. This contradicts the fact that φ is big. □

As promised, we can now show:

THEOREM 3.4. *A countable theory T which is categorical in an uncountable cardinal is ω-stable, hence totally transcendental, hence κ-stable for any infinite κ.*

PROOF. Let N be a model and assume that there is a countable subset $A \subseteq N$ such that there are uncountably many types over A. We may assume that all these types are realised in N (see exercise 5 below), so let $(b_i)_{i \in I}$ be a sequence of ω_1-many elements from N realising different types over A. First choose an elementary substructure M_0 of N of cardinality ω_1 which contains both A and the b_i, and then choose an elementary extension M of M_0 of cardinality κ. The model M is of cardinality κ and realises uncountably many types over the countable set A. But Theorem 2.1 implies that T also has a model of cardinality κ in which this is not the case. So T is not κ-categorical. □

2. More on saturated models

The result from the previous section can be used to characterise κ-categorical theories in terms of saturated models. But first we need a lemma:

LEMMA 3.5. *If T is a countable theory which is κ-stable, then for all regular $\lambda \leq \kappa$ there is a model of cardinality κ which is λ-saturated.*

PROOF. We constuct a sequence $(M_\alpha : \alpha \in \lambda)$ of models of T of cardinality κ. We start with any model M_0 of cardinality κ of T; at limit stages we take the colimit and at successor stages we take a model $M_{\alpha+1}$ which realises all types in $S(M_\alpha)$. This we can do with a model of cardinality κ since $|S(M_\alpha)| \leq \kappa$. The colimit of the entire chain will be λ-saturated. □

THEOREM 3.6. *A countable theory T is κ-categorical if and only if all models of cardinality κ are κ-saturated.*

PROOF. Note that we already proved this result for $\kappa = \omega$ (if not, do exercise 8) and that we also know that any two κ-saturated models of cardinality κ are isomorphic. So we only need to show that if T is κ-categorical for some uncountable cardinal κ, then its unique model of cardinality κ is κ-saturated.

But then T is ω-stable, hence totally transcendental, hence κ-stable. So by the lemma the unique model of T of cardinality κ is μ^+-saturated for all $\mu < \kappa$. So this model is κ-saturated. □

3. Exercises

EXERCISE 3. Show that the theory DLO of dense linear orders without endpoints is not ω-stable.

EXERCISE 4. For the algebraists among us: show by hand (that is, without using any of the results from this chapter) that the theory of algebraically closed fields in characteristic 0 is ω-stable.

EXERCISE 5. Let κ be an infinite cardinal. Show that a theory T is κ-stable if in each model M of T there are at most κ many 1-types over each set of parameters of size at most κ from M.

EXERCISE 6. Let κ be an infinite cardinal. Show that a theory T is κ-stable if in each model M of T there are at most κ many 1-types realised in M over each set of parameters of size at most κ from M. (Remember that all theories are assumed to be complete!)

EXERCISE 7. If T is an L-theory and K is a sublanguage of L, then the *reduct* $T \restriction K$ is the set of all K-sentences which follow from T. Show that T is totally transcendental if and only if $T \restriction K$ is ω-stable for any at most countable $K \subseteq L$.

EXERCISE 8. Show that if T is a countable ω-categorical theory, then its unique model of cardinality ω is ω-saturated.

CHAPTER 4

Totally transcendental theories

In the previous chapter we have established that countable theories which are categorical in some uncountable cardinal are totally transcendental. For this reason we now embark on a study of these totally transcendental theories. In fact, for the proof of Morley's theorem we need two results about these theories, one on atomic extensions and one on indiscernibles. The second is the most difficult of the two and requires the notions of Morley rang and degree; we will introduce these notions and prove the second result in the next chapter. This chapter will devoted to the proof of the first result:

THEOREM 4.1. *Assume T is a totally transcendental theory. If $M \models T$ and $B \subseteq M$, then there exists $A \preceq M$ such that $B \subseteq A$ and A is atomic over B.*

Before we explain the terminology, let us first recall that if $p(\overline{x})$ is an isolated type in the type space of some theory T, there is formula $\varphi(\overline{x})$ such that $p(\overline{x})$ is the sole element of $[\varphi(\overline{x})]$. This means that $\varphi(\overline{x}) \in p(\overline{x})$ and every formula $\psi(\overline{x})$ in $p(\overline{x})$ is a consequence over T of $\varphi(\overline{x})$. In this case, we will say that $\varphi(\overline{x})$ *isolates* $p(\overline{x})$ and that the formula $\varphi(\overline{x})$ is *isolating* (sometimes also called *principal* or *complete*).

THEOREM 4.2. *If T is a totally transcendental theory, then isolated types are dense in the type spaces of T. Therefore, if T is countable, T has a prime model.*

PROOF. If isolated types are not dense, then there is a consistent $\varphi(\overline{x})$ which is not a consequence of an isolating formula. Call such a formula *perfect*. Since perfect formulas are not isolating, they can be "decomposed" into two consistent formulas which are jointly inconsistent. These have to be perfect as well, leading to a binary tree of consistent formulas. □

We can now explain what we mean by atomic extensions:

DEFINITION 4.3. Let A be a model and $B \subseteq A$. An element $a \in A$ is *atomic* over B if the type $\mathrm{tp}(a/B)$ is isolated. If each element $a \in A$ is atomic over B, we say that A is *atomic* over B. We will say that A is *constructible* over B, if there is an ordinal γ and an enumeration $(a_\alpha)_{\alpha<\gamma}$ of A, such that each a_α is atomic over $B \cup A_\alpha$, where $A_\alpha = \{a_\beta : \beta < \alpha\}$.

As a first step towards proving Theorem 4.1, we will first establish the corresponding result for constructible extensions.

LEMMA 4.4. *Assume T is totally transcendental. If $M \models T$ and $B \subseteq M$, then there exists $A \preceq M$ such that $B \subseteq A$ and A is constructible over B.*

PROOF. First note that if T is totally transcendental, and A is any subset of a model M of T, then $T_A = \{\varphi \in L_A : M \models \varphi\}$ is totally transcendental as well. Hence isolated types are dense in the type spaces of T_A by Theorem 4.2.

Let us call an enumeration $(a_\alpha)_{\alpha<\gamma}$ of some subset of M a *construction*, if each a_α is atomic over $B \cup A_\alpha$. Zorn's Lemma allows us to find a maximal construction $(a_\alpha)_{a<\gamma}$ which cannot be prolonged by an element $a_\gamma \in M$. Writing $A = A_\gamma$ for the set of elements of this maximal construction, we clearly have that $B \subseteq A$ and that A is closed under all the functions in M. So it remains to show that A is the universe of an elementary substructure of M. For this we employ the Tarski-Vaught Test.

So assume $\varphi(x)$ is an L_A-formula and $M \models \exists x\, \varphi(x)$. Since isolated types over A are dense, there is an isolated $p(x) \in S(T_A)$ with $\varphi(x) \in p(x)$. Let a be a realisation of $p(x)$ in M. If $a \notin A$, then we could prolong our construction by $a_\gamma = a$; thus $a \in A$ and $\varphi(x)$ is realised in A. □

This means that we would obtain a proof of Theorem 4.1 if we could show that constructible extensions are atomic. This is true, but to prove that we need another lemma:

LEMMA 4.5. *Let $\overline{a}$ and $\overline{b}$ be two finite tuples of elements of a structure M. Then* $\mathrm{tp}(\overline{a}\overline{b})$ *is isolated if and only if* $\mathrm{tp}(\overline{a}/\overline{b})$ *and* $\mathrm{tp}(\overline{b})$ *are isolated.*

PROOF. First assume that $\varphi(\overline{x},\overline{y})$ isolates $\mathrm{tp}(\overline{a},\overline{b})$. Then $\varphi(\overline{x},\overline{b})$ isolates $\mathrm{tp}(\overline{a}/\overline{b})$ and we claim $\exists\overline{x}\,\varphi(\overline{x},\overline{y})$ isolates $p(\overline{y}) = \mathrm{tp}(\overline{b})$: we have $\exists\overline{x}\,\varphi(\overline{x},\overline{y}) \in p(\overline{y})$ and if $\sigma(\overline{y}) \in p(\overline{y})$, then $M \models \forall\overline{x},\overline{y}\,(\,\varphi(\overline{x},\overline{y}) \to \sigma(\overline{y})\,)$ and hence $M \models \forall\overline{y}\,(\,\exists\overline{x}\,\varphi(\overline{x},\overline{y}) \to \sigma(\overline{y})\,)$.

Conversely, suppose $\rho(\overline{x},\overline{b})$ isolates $\mathrm{tp}(\overline{a}/\overline{b})$ and $\sigma(\overline{y})$ isolates $p(\overline{y}) = \mathrm{tp}(\overline{b})$. Then $\rho(\overline{x},\overline{y}) \wedge \sigma(\overline{y})$ isolates $\mathrm{tp}(\overline{a}\overline{b})$. For if $\varphi(\overline{x},\overline{y}) \in \mathrm{tp}(\overline{a}\overline{b})$, then $\varphi(\overline{x},\overline{b})$ belongs to $\mathrm{tp}(\overline{a}/\overline{b})$ and $M \models \forall\overline{x}\,(\,\rho(\overline{x},\overline{b}) \to \varphi(\overline{x},\overline{b})\,)$. Hence $\forall\overline{x}\,(\,\rho(\overline{x},\overline{y}) \to \varphi(\overline{x},\overline{y})) \in p(\overline{y})$ and so it follows that $M \models \forall\overline{y}\,(\,\sigma(\overline{y}) \to \forall\overline{x}\,(\,\rho(\overline{x},\overline{y}) \to \varphi(\overline{x},\overline{y})\,)$. Thus $M \models \forall\overline{x},\overline{y}\,(\,\rho(\overline{x},\overline{y}) \wedge \sigma(\overline{y}) \to \varphi(\overline{x},\overline{y})\,)$. □

LEMMA 4.6. *Let A be an L-structure and $B \subseteq A$. If A is constructible over B, then it is also atomic over B.*

PROOF. Without loss of generality, we may assume that the structure A contains names for all the elements of B. So let A be a constructible over B and let $\overline{a}$ be a tuple from A. We have to show that $\overline{a}$ is atomic.

We can clearly assume that the elements of $\overline{a}$ are pairwise distinct and do not belong to B. In addition, we can permute the elements of $\overline{a}$ so that

$$\overline{a} = a_\alpha\overline{b}$$

for some tuple $\overline{b} \in A_\alpha$. By assumption, there is an L-formula $\varphi(x,\overline{y})$ and a tuple $\overline{c}$ from A_α such that $\varphi(x,\overline{c})$ isolates the type of a_α over A_α. In particular, a_α is atomic over $\overline{b}\overline{c}$. Using induction, we know that $\overline{b}\overline{c}$ is atomic, so the previous lemma gives us that both $a_\alpha\overline{b}\overline{c}$ and $\overline{a} = a_\alpha\overline{b}$ are atomic. □

This completes the proof of Theorem 4.1, and hence of the main result of this chapter.

1. Exercises

EXERCISE 9. Let A be a model of some theory T and B be a subset of A. We will say that A is *prime* over B, if any partial elementary map $B \to M$ extends to an elementary map $A \to M$.

(i) Show that if A is constructible over B, then A is also prime over B.

(ii) Show that if T is totally transcendental and A is prime over B, then A is also atomic over B.

CHAPTER 5

Morley rank and degree

The aim of this chapter is to give a proof of:

THEOREM 5.1. *Assume T is a countable and totally transcendental theory, and suppose $A \models T$ and $C \subseteq A$. If A is uncountable and $|C| < |A|$, then there is a nonconstant sequence $(a_k : k \in \mathbb{N})$ of indiscernibles in $(A, c)_{c \in C}$.*

(Here $(A, c)_{c \in C}$ is result of adding constants to the model A for all the elements in C.) To prove this result, we need to introduce the concepts of Morley rank and degree; these concepts make sense for any theory T, but they work best in case T is totally transcendental, as we shall see.

1. Morley rank

DEFINITION 5.2. Suppose A is an ω-saturated model, $\varphi(x)$ is an L_A-formula, and α is an ordinal. We define $\mathrm{RM}_x(A, \varphi(x)) \geq \alpha$ by induction on α:

(1) $\mathrm{RM}_x(A, \varphi(x)) \geq 0$ if $A \models \exists x\, \varphi(x)$;
(2) $\mathrm{RM}_x(A, \varphi(x)) \geq \alpha + 1$ if there is a sequence $(\varphi_k(x) \colon k \in \mathbb{N})$ of L_A-formulas such that
   (a) $A \models \forall x\, (\, \varphi_k(x) \to \varphi(x)\,)$ for all $k \in \mathbb{N}$;
   (b) $A \models \forall x\, \neg(\, \varphi_k(x) \wedge \varphi_l(x)\,)$ for all distinct $k, l \in \mathbb{N}$;
   (c) $\mathrm{RM}_x(A, \varphi_k(x)) \geq \alpha$ for all $k \in \mathbb{N}$;
(3) for λ a limit ordinal, $\mathrm{RM}_x(A, \varphi(x)) \geq \lambda$ if $\mathrm{RM}_x(A, \varphi(x)) \geq \alpha$ for all $\alpha < \lambda$.

REMARK 5.3. In this definition it has to be understood that x could be a tuple as well.

LEMMA 5.4. *Suppose A is an ω-saturated model and $\varphi(x)$ is an L_A-formula. Let S be the class of ordinals α such that $\mathrm{RM}_x(A, \varphi(x)) \geq \alpha$ holds. Then exactly one of the following alternatives holds:*

(1) *S is empty;*
(2) *S is the class of all ordinals;*
(3) *$S = \{\alpha \colon \alpha \leq \gamma\}$ for some ordinal γ.*

PROOF. This really amounts to showing that $\mathrm{RM}_x(A, \varphi(x)) \geq \alpha$ and $\alpha > \beta \geq 0$ imply $\mathrm{RM}_x(A, \varphi(x)) \geq \beta$. We prove this by induction on α and β. The cases where α or β is a limit ordinal are easy, so assume $\mathrm{RM}_x(A, \varphi(x)) \geq \alpha + 1$ and $\alpha + 1 > \beta + 1$ (so $\alpha > \beta$). The first assumption implies that there is a sequence $(\varphi_k(x) \colon k \in \mathbb{N})$ with $\mathrm{RM}_x(A, \varphi_k(x)) \geq \alpha$. But then $\mathrm{RM}_x(A, \varphi_k(x)) \geq \beta$ and hence $\mathrm{RM}_x(A, \varphi(x)) \geq \beta + 1$, as desired. □

DEFINITION 5.5. Let A be an ω-saturated model and let $\varphi(x)$ be an L_A-formula. If the third alternative from Lemma 5.4 holds, we call $\varphi(x)$ *ranked with Morley rank* γ and we will write $\mathrm{RM}_x(A, \varphi(x))$ for γ.

Note that in Lemma 5.4 the first alternative holds only when $\varphi(x)$ cannot be realised in A. So if a formula $\varphi(x)$ is realised in A, but has no Morley rank, then this must mean that $\mathrm{RM}_x(A, \varphi(x)) \geq \alpha$ holds for all ordinals α. (Some authors write $\mathrm{RM}_x(A, \varphi(x)) = \infty$ in this case.)

A natural question is: if $\varphi(x)$ is an L_A-formula, but A is not ω-saturated, can we still define the Morley rank of A? Of course, we could take an elementary extension B of A which is ω-saturated and compute the Morley rank of φ there, but right now it is not clear that this does not depend on our choice B. As we will see, it does not depend on this choice and therefore we can safely define the Morley rank of $\varphi(x)$ in this way. In fact, something stronger is true:

Lemma 5.6. *Let A be a model and $\varphi(x, y)$ be an L-formula. If a is a finite tuple of elements of A and B is an ω-saturated elementary extension of A, then the value of $\mathrm{RM}_x(B, \varphi(x, a))$ depends only on $\mathrm{tp}_A(a)$.*

Proof. It suffices to prove that the truth value of $\mathrm{RM}_x(B, \varphi(x, a)) \geq \alpha$ only depends on the type of a. We prove this by induction on α; the case that $\alpha = 0$ or a limit ordinal is trivial. So assume the statement holds for all $\alpha < \beta + 1$.

For $j = 1, 2$, let A_j be a model of T and a_j be finite tuples from A_j with $\mathrm{tp}_{A_1}(a_1) = \mathrm{tp}_{A_2}(a_2)$ and B_j be ω-saturated elementary extensions of A_j. We assume $\mathrm{RM}_x(B_1, \varphi(x, a_1)) \geq \beta + 1$ and need to prove $\mathrm{RM}_x(B_2, \varphi(x, a_2)) \geq \beta + 1$. This assumption yields a sequence of formulas $(\varphi_k(x, b_k) : k \in \mathbb{N})$ to witness that $\mathrm{RM}_x(B_1, \varphi(x, a_1)) \geq \beta + 1$, that is,

(1) $B_1 \models \forall x\, (\, \varphi_k(x, b_k) \to \varphi(x, a_1)\,)$ for all $k \in \mathbb{N}$;
(2) $B_1 \models \forall x\, \neg(\, \varphi_k(x, b_k) \wedge \varphi_l(x, b_l)\,)$ for all distinct $k, l \in \mathbb{N}$;
(3) $\mathrm{RM}_x(B_1, \varphi_k(x, b_k)) \geq \beta$ for all $k \in \mathbb{N}$.

Since B_2 is ω-saturated and $\mathrm{tp}_{B_1}(a_1) = \mathrm{tp}_{B_2}(a_2)$, we may construct inductively a sequence $(c_k : k \in \mathbb{N})$ of finite tuples from B_2 such that for all $k \in \mathbb{N}$

$$\mathrm{tp}_{B_2}(a_2 c_0 \dots c_k) = \mathrm{tp}_{B_1}(a_1 b_0 \dots b_k).$$

It follows that

(1) $B_2 \models \forall x\, (\, \varphi_k(x, c_k) \to \varphi(x, a_2)\,)$ for all $k \in \mathbb{N}$;
(2) $B_1 \models \forall x\, \neg(\, \varphi_k(x, c_k) \wedge \varphi_l(x, c_l)\,)$ for all distinct $k, l \in \mathbb{N}$;
(3) $\mathrm{RM}_x(B_2, \varphi_k(x, c_k)) \geq \beta$ for all $k \in \mathbb{N}$.

(Statements (1) and (2) are immediate; for (3) use the induction hypothesis.) We conclude $\mathrm{RM}_x(B_2, \varphi_k(x, a_2)) \geq \beta + 1$, as desired. □

The following theorem makes clear why the concept of Morley rank is a tool which is especially suited for analysing totally transcendental theories.

Theorem 5.7. *The following are equivalent for a theory T:*

(1) *T is totally transcendental.*
(2) *if $A \models T$ and $\varphi(x)$ is an L_A-formula which is realised in A, then $\varphi(x)$ is ranked.*

Proof. (1) ⇒ (2): Let A be a model of T and let $\varphi(x)$ be an L_A-formula which is realised, but not ranked (that is, $\mathrm{RM}(\varphi(x)) \geq \alpha$ holds for all α). We may assume that A is ω-saturated. Since the formulas from L_A form a set, there is an ordinal α such that there are no ranked formulas whose Morley rank is $\geq \alpha$. So because $\mathrm{RM}(\varphi(x)) \geq \alpha + 1$, there must be contradictory formulas $\psi_1(x)$ and $\psi_2(x)$ with $\mathrm{RM}(\psi_i(x)) \geq \alpha$ and $A \models \psi_i(x) \to \varphi(x)$. So $\varphi(x) \wedge \psi_1(x)$ and $\varphi(x) \wedge \psi_2(x)$ are both realised in A and unranked. Continuing in this way we create a binary tree of consistent formulas in A, so T is not totally transcendental.

(2) $\Rightarrow$ (1): Conversely, if T is not totally transcendental, then there is a model A together with a binary tree $(\varphi_s : s \in \{0,1\}^*)$ of L_A-formulas satisfying conditions (1)-(3) from Definition 3.1. We claim that none of the formulas φ_s can be ranked. For if there would a formula φ_s with a rank, there would have to be a φ_s which has minimal rank α. But then consider $\varphi_{s0}, \varphi_{s10}, \varphi_{s110}, \ldots$: these formulas are realised and mutually contradictory, and imply φ_s in A. But then at least one of these formulas must have a Morley rank which is strictly smaller than α, contradicting the minimality of α. □

In the sequel we will need the following computation rules for the Morley rank:

Lemma 5.8. *Let A be an ω-saturated model and let $\varphi(x), \psi(x)$ be L_A-formulas.*

1. $\mathrm{RM}_x(A, \varphi(x)) = 0$ *iff the number of tuples $u \in A$ for which $A \models \varphi(u)$ is finite and* > 0.
2. *if $A \models \varphi(x) \to \psi(x)$, then* $\mathrm{RM}_x(A, \varphi(x)) \leq \mathrm{RM}_x(A, \psi(x))$.
3. $\mathrm{RM}_x(A, \varphi(x) \vee \psi(x)) = \max(\mathrm{RM}_x(A, \varphi(x)), \mathrm{RM}_x(A, \psi(x)))$.

We will not prove this lemma; in fact, it will be exercise 10.

The notion of Morley rank can be extended to types, as follows:

Definition 5.9. Let M be a model and A be a subset of M. We will call a type $p(x)$ over A *ranked*, if it contains at least one ranked formula. In that case, the *Morley rank of p* is the least Morley rank of a formula in $p(x)$.

A key property of ranked types is that are completely determined by a single element: they are like isolated types in that respect. To show this, we need the notion of Morley degree.

2. Morley degree

The definition of Morley degree relies on the following lemma:

Lemma 5.10. *Let A be an ω-saturated model and $\varphi(x)$ be a ranked L_A-formula with Morley rank α. There exists a finite bound on the integers k such that there is a sequence of L_A-formulas $(\varphi_j(x) \colon 0 \leq j < k)$ such that*

1. $\mathrm{RM}_x(A, \varphi_j(x)) = \mathrm{RM}_x(A, \varphi(x))$ *for all $j < k$;*
2. $A \models (\varphi_j(x) \to \varphi(x))$ *for all $j < k$;*
3. $A \models \neg(\varphi_i(x) \wedge \varphi_j(x))$ *for distinct $i, j < k$.*

Moreover, there is a sequence $(\varphi_j(x) \colon 0 \leq j < k)$ realising the maximal such k for which $A \models \varphi(x) \leftrightarrow \bigvee_j \varphi_j(x)$.

Proof. We will construct a sequence of L_A-formulas $(\varphi_j(x) \colon 0 \leq j < k)$ satisfying (1)-(3) which realises the maximal possible k; to this purpose, we will create a binary tree of L_A-formulas, each having Morley rank α. We start by putting $\varphi_{<>} = \varphi(x)$. If φ_s has been constructed, we check whether there is a formula ψ such that both $\varphi \wedge \psi$ and $\varphi \wedge \neg\psi$ have Morley rank α. If so, we put $\varphi_{s0} = \varphi \wedge \psi$ and $\varphi_{s1} = \varphi \wedge \neg\psi$ for some such ψ. Otherwise we stop.

The resulting tree has to be finite: for otherwise it would have (by König's Lemma, see the Appendix) an infinite branch B. But then $\varphi_s \wedge \neg\varphi_t$ for any two $s, t \in B$ such that t is an immediate successor of s would be an infinite sequence witnessing that the Morley rank of φ is $\geq \alpha + 1$.

So let L be the collection of leaves of the tree. Then $(\varphi_s : s \in L)$ is a sequence satisfying (1)-(3): in fact, $\varphi \leftrightarrow \bigvee_{s\in L} \varphi_s$. We claim it realises the maximal possible k.

For suppose $(\psi_j(x) \colon 0 \leq j < k)$ is another such sequence satisfying (1)-(3) and $k > |L|$. Then, for a fixed $s \in L$, there can be at most one $i < k$ such that $\varphi_s \wedge \psi_i$ has Morley rank α: for the formulas $\psi_i(x)$ and $\psi_j(x)$ are contradictory whenever i and j are distinct, and $s \in L$ is a leaf, so cannot be split into two contradictory formulas of rank α. But then it follows from the Pigeonhole Principle and the fact that $k > |L|$, that there is a $j < k$ such that $\psi_j \wedge \varphi_s$ has rank $< \alpha$ for all $s \in L$. But as ψ_j is equivalent to the disjunction of all formulas $\psi_j \wedge \varphi_s$, it follows that ψ_j must itself have Morley rank $< \alpha$. Contradiction! □

Definition 5.11. If A is ω-saturated and $\varphi(x)$ is a ranked L_A-formula, the greatest integer whose existence we just proved is called the *Morley degree* of $\varphi(x)$ and it is denoted by $dM(\varphi(x))$.

If $\varphi(x)$ is a ranked L_A-formula, but A is not ω-saturated, we can still define the Morley degree of $\varphi(x)$ by embedding A into an ω-saturated elementary extension B of A and computing the Morley degree in B. Similarly as in Lemma 5.6, one can show that this does not depend on the choice of B; in fact, that the Morley degree of $\varphi(x, a)$, where $\varphi(x, y)$ is an L-formula and a is a tuple from A, depends only on $\mathrm{tp}_A(a)$.

The notion of Morley degree has the following properties:

Lemma 5.12. *Let A be an ω-saturated model and let $\varphi(x)$ and $\psi(x)$ be ranked L_A-formulas.*

(1) *If $dM(\varphi(x)) = d$ and this is witnessed by the sequence $(\varphi_j(x) \colon 0 \leq j < d)$, then each $\varphi_j(x)$ has Morley degree 1.*
(2) *If $\mathrm{RM}_x(A, \varphi(x)) = \mathrm{RM}_x(A, \psi(x))$ and $A \models \varphi(x) \to \psi(x)$, then $dM(\varphi(x)) \leq dM(\psi(x))$.*
(3) *If $\mathrm{RM}_x(A, \varphi(x)) = \mathrm{RM}_x(A, \psi(x))$, then $dM(\varphi(x) \vee \psi(x)) \leq dM(\varphi(x)) + dM(\psi(x))$, with equality if $A \models \neg(\,\varphi(x) \wedge \psi(x)\,)$.*
(4) *If $\mathrm{RM}_x(A, \varphi(x)) < \mathrm{RM}_x(A, \psi(x))$, then $dM(\varphi(x) \vee \psi(x)) = dM(\psi(x))$.*

Again, we leave the proof of this lemma as an exercise.

The notion of Morley degree can be extended to types, as follows:

Definition 5.13. Let M be a model and A be a subset of M, and assume $p(x)$ is a ranked type over A with Morley rank α. The *Morley degree of p* is the least Morley degree of a formula with Morley rank α in $p(x)$.

We now have the following result, as promised:

Theorem 5.14. *Let M be a model and A be a subset of M, and let $p(x)$ be a ranked type over A with Morley rank α and Morley degree d. Then $p(x)$ contains a formula $\varphi(x)$ with Morley rank α and Morley degree d, by definition, and any such formula $\varphi(x)$ determines the type $p(x)$ in that $p(x)$ consists exactly of the L_A-formulas $\psi(x)$ such that $\mathrm{RM}(\psi(x) \wedge \varphi(x)) = \mathrm{RM}(\varphi(x))$.*

Proof. We may assume that M is ω-saturated. Now, if $\psi(x)$ is any formula in $p(x)$, also $\psi(x) \wedge \varphi(x) \in p(x)$ and hence $\mathrm{RM}(\psi(x) \wedge \varphi(x)) \geq \mathrm{RM}(\varphi(x))$ by choice of $\varphi(x)$. Hence $\mathrm{RM}(\psi(x) \wedge \varphi(x)) = \mathrm{RM}(\varphi(x))$.

Conversely, suppose $\psi(x)$ is any L_A-formula with $\mathrm{RM}(\psi(x) \wedge \varphi(x)) = \mathrm{RM}(\varphi(x))$ and $dM(\psi(x) \wedge \varphi(x)) = dM(\varphi(x))$. By way of contradiction, if $\psi(x) \notin p(x)$, then $\neg\psi(x) \in p(x)$ and $\neg\psi(x) \wedge \varphi(x) \in p(x)$. But then $\mathrm{RM}(\varphi(x)) = \mathrm{RM}(\neg\psi(x) \wedge \varphi(x))$ and $dM(\varphi(x)) \geq dM(\psi(x) \wedge \varphi(x)) + dM(\neg\psi(x) \wedge \varphi(x)) > dM(\neg\psi(x) \wedge \varphi(x))$, which contradicts the choice of $\varphi(x)$. □

3. Morley sequence

In this section, we will prove the main result of this chapter. But before we can do this, we need one more definition and one more technical lemma.

Definition 5.15. Let A be a model and $C \subseteq A$, and let $\varphi(x)$ be a ranked L_C-formula with Morley rank α and Morley degree d. A sequence $(a_\gamma : \gamma < \delta)$ of elements from A indexed by an infinite ordinal δ is a *Morley sequence* over $\varphi(x)$, if for every $\gamma < \delta$ we have that $A \models \varphi(a_\gamma)$ and that $\mathrm{tp}(a_\gamma / C \cup A_\gamma)$ has Morley rank α and Morley degree d.

Lemma 5.16. *Let A be a model and $C \subseteq A$, and let $\varphi(x)$ be a ranked L_C-formula with Morley rank α and Morley degree d. If $(a_\gamma : \gamma < \delta)$ is a Morley sequence over $\varphi(x)$, then it is a sequence of indiscernibles over C.*

Proof. We prove by induction on n that whenever $\gamma_0 < \ldots < \gamma_n$, then $\mathrm{tp}(a_{\gamma_0}, \ldots, a_{\gamma_n}/C) = \mathrm{tp}(a_0, \ldots, a_n/C)$.

First consider $n = 0$. We have $A \models \varphi(a_\gamma)$, so the type $\mathrm{tp}(a_\gamma/C)$ contains $\varphi(x)$; also, its rank is α and degree is d, hence $\mathrm{tp}(a_\gamma/C)$ is completely determined by $\varphi(x)$, as in Theorem 5.14. But as the way in which it is determined by this formula does not depend on γ, all $\mathrm{tp}(a_\gamma/C)$ have to be identical.

So suppose the statement is true for n. Both $\mathrm{tp}(a_{\gamma_{n+1}}/C \cup \{a_{\gamma_0}, \ldots, a_{\gamma_n}\})$ and $\mathrm{tp}(a_{n+1}/C \cup \{a_0, \ldots, a_n\})$ have rank α and degree d and contain $\varphi(x)$, so we have:

(i) $A \models \psi(a_{\gamma_0}, \ldots, a_{\gamma_n}, a_{\gamma_{n+1}})$ iff $\psi(a_{\gamma_0}, \ldots, a_{\gamma_n}, x) \wedge \varphi(x)$ has Morley rank α.
(ii) $A \models \psi(a_0, \ldots, a_n, a_{n+1})$ iff $\psi(a_0, \ldots, a_n, x) \wedge \varphi(x)$ has Morley rank α.

The induction hypothesis gives us $\mathrm{tp}(a_{\gamma_0}, \ldots, a_{\gamma_n}/C) = \mathrm{tp}(a_0, \ldots, a_n/C)$, so the right hand sides in (i) and (ii) are equivalent by Lemma 5.6. Hence

$$\mathrm{tp}(a_{\gamma_0}, \ldots, a_{\gamma_n}, a_{\gamma_{n+1}}/C) = \mathrm{tp}(a_0, \ldots, a_n, a_{n+1}/C),$$

as desired. □

Theorem 5.17. *Assume T is a countable and totally transcendental theory, and suppose $A \models T$ and $C \subseteq A$. If A is uncountable and $|C| < |A|$, then there is a nonconstant sequence $(a_k : k \in \mathbb{N})$ of indiscernibles in $(A, c)_{c \in C}$.*

Proof. We may assume C is infinite. Write $\lambda = |C|$. The formula $x = x$ is satisfied by $> \lambda$ many elements, so choose an L_A-formula $\varphi(x)$ that is satisfied by $> \lambda$ many elements and has minimum possible Morley rank α and Morley degree d (in lexicographic order). Note that $\alpha > 0$ since $\varphi(x)$ is satisfied by infinitely many elements. By adding finitely many elements to C we may assume that $\varphi(x)$ is an L_C-formula.

We will construct by induction on k a sequence $(a_k : k \in \mathbb{N})$ of elements of A that satisfy $\varphi(x)$ and such that $\mathrm{tp}_A(a_k/C \cup \{a_0, \ldots, a_{k-1}\})$ has Morley rank α and Morley degree d. Hence it will be an indiscernible sequence by the previous lemma.

First we claim that there is an a_0 with this property. For if no such element would exist, we would have that Morley rank and degree of $\text{tp}_A(a/C)$ is $< (\alpha, d)$ for all $a \in A$ satisfying $\varphi(x)$. So each $a \in A$ which satisfies $\varphi(x)$ also satisfies an L_C-formula $\psi_a(x)$ with Morley rank and degree $< (\alpha, d)$. But since there are at most λ many L_C-formulas and more than λ many a satisfying $\varphi(x)$, there must be a formula with Morley rank and degree $< (\alpha, d)$ satisfied by more than λ many a. Contradiction!

The construction of a_k given $a_0, \ldots, a_{k-1}$ is similar. In fact, we can give the same argument as in the previous paragraph, replacing C by $C \cup \{a_0, \ldots, a_{k-1}\}$. □

4. Exercises

By $\text{RM}(T)$ and $dM(T)$, the Morley rank and Morley degree of a theory T, we will mean the Morley rank and Morley degree of the formula $x = x$ over any (ω-saturated) model of T.

Exercise 10. Prove Lemma 5.8.

Exercise 11. Prove Lemma 5.12.

Exercise 12. Let A be an ω-saturated model and $\varphi(x)$ be a ranked L_A-formula. If $\text{RM}_x(A, \varphi(x)) \geq \alpha$, then there exists an L_A-formula $\psi(x)$ such that $A \models \psi(x) \to \varphi(x)$ and $\text{RM}_x(A, \psi(x)) = \alpha$.

Exercise 13. Let A be an L-structure, let $\varphi(\overline{x})$ be an L_A-formula and let $t(\overline{x})$ be an L-term. Show that the formulas $(\varphi(\overline{x}) \wedge y = t(\overline{x}))$ and $\varphi(x)$ have the same Morley rank. (Here y is a single, new variable. The Morley rank of $\varphi(\overline{x})$ is taken with respect to the variables $\overline{x}$ and the Morley rank of $(\varphi(\overline{x}) \wedge y = t(\overline{x}))$ is taken with respect to the variables $\overline{x}, y$.)

Exercise 14. Let L be the language consisting of unary predicate symbols $P_1, ..., P_n$. Let T be the L-theory whose axioms express that the sets $P_1, ..., P_n$ are infinite and that they form a partition of the underlying set of the L-structure being considered. Show that T admits quantifier elimination and is complete. Show that T has Morley rank 1 and Morley degree n.

Exercise 15. Let A be an ω-saturated model and $p(x)$ be a ranked type over A. Show that $p(x)$ has Morley degree 1.

Exercise 16. (This is Marker 6.6.17.)

(i) Let $L = \{E\}$, where E is a binary relation symbol. Let T be the theory of an equivalence relaton with infinitely many classes, each of which is infinite. Show that $\text{RM}(T) = 2$.
(ii) Let $L = \{P_0, P_1, \ldots\}$, where each P_i is a unary predicate. Let T be the theory that asserts $P_0 \supseteq P_1 \supseteq \ldots$, with the complement of P_0 infinite and $P_n \setminus P_{n+1}$ infinite for each n. Show that $\text{RM}(T) = 2$.
(iii) For each $n < \omega$, give an example of a theory with $\text{RM}(T) = n$.

Exercise 17. Let T be a theory in a language L. If a type p over T has a Morley rank, then $\text{RM}(p) < |L|^+$. Hence, if T is totally transcendental, we have $\text{RM}(T) < |T|^+$.

Exercise 18. Let A be $(\mathbb{Z}, S, <)$, the integers together with the successor function and the usual order, and put $T = \text{Th}(A)$.

(i) Show that T has quantifier elimination.
(ii) Show that T has a countable ω-saturated model B. Describe B concretely.

(iii) What is the Morley rank of T? And what would happen if we would compute the Morley rank of $x = x$ in A, as in Definition 5.2, ignoring the fact that A is not ω-saturated?

EXERCISE 19. (This is Marker 6.6.19.) Let X be a compact Hausdorff space. For every subset A of X we define

$$\Gamma(A) = \{x \in A : x \text{ is not isolated } A\}.$$

For α an ordinal, we inductively define $\Gamma^\alpha(X)$ as follows:

$$\begin{aligned} \Gamma^0(X) &= X, \\ \Gamma^{\alpha+1}(X) &= \Gamma(\Gamma^\alpha(X)), \\ \Gamma^\lambda(X) &= \bigcap_{\alpha<\lambda} \Gamma^\alpha(X), \quad \text{if } \lambda \text{ is a limit ordinal} \end{aligned}$$

The chain $\Gamma^\alpha(X)$ is monotonically decreasing and hence there must an ordinal δ such that $\Gamma^\alpha(X) = \Gamma^\delta(X)$ for all $\alpha > \delta$. We will write $\Gamma^\infty(X)$ for $\Gamma^\delta(X)$.

(i) Prove that $\Gamma^\infty(X)$ is a closed subset of X without isolated points.
(ii) Suppose X has a countable basis. Prove that there is a countable α such that $\Gamma^\alpha(X) = \Gamma^\infty(X)$.
(c) Prove that if X has a countable basis, then $\Gamma^\infty(X) = \emptyset$ of $|\Gamma^\infty(X)| = 2^\omega$.
(d) Suppose T is totally transcendental and M is a model of T. We will say that $p \in S_n(T_M)$ has *Cantor-Bendixson rank* α, whenever $p \in \Gamma^\alpha(S_n(T_M)) \setminus \Gamma^{\alpha+1}(S_n(T_M))$. Show that every type has a Cantor-Bendixson rank and that the Cantor-Bendixson rank is exactly the Morley rank.

CHAPTER 6

Proof of Morley's Theorem

By now we have assembled all the ingredients we need for proving Morley's Theorem.

THEOREM 6.1. *If T is a countable theory which is κ-categorical for some uncountable cardinal κ, then T is λ-categorical for all uncountable cardinals λ.*

PROOF. In view of Theorem 3.4 and Theorem 3.6, it is sufficient to prove the following statement:

> Let T be a countable and totally transcendental theory, and assume κ is an uncountable cardinal. If every model of T of cardinality κ is κ-saturated, then every uncountable model of T is saturated.

We prove the contraposition. So suppose T is a totally transcendental theory in a countable language L and assume T has a model A of cardinality λ that is not λ-saturated. Our goal will be to construct a model of cardinality κ that is not κ-saturated.

Our assumption means that there is a subset C of A of cardinality $< \lambda$ and a type $p(x)$ over C which is not realized in $(A, c)_{c \in C}$. Theorem 5.17 tells us that there is a nonconstant sequence $(a_k : k \in \mathbb{N})$ of indiscernibles in $(A, c)_{c \in C}$. Write $I = \{a_k : k \in \mathbb{N}\}$ and note that:

> (†) For each $L_{C \cup I}$-formula $\varphi(x)$ that is satisfiable in $(A, a)_{a \in C \cup I}$ there exists $\psi(x) \in p(x)$ such that $\varphi(x) \wedge \neg\psi(x)$ is satisfiable in $(A, a)_{a \in C \cup I}$.

(For otherwise $p(x)$ would be realised in $(A, a)_{a \in C}$.) We claim that (†) holds also for some type over a countable set C. Indeed, let C_0 be any countable subset of C. For each $L_{C_0 \cup I}$-formula $\varphi(x)$ that is satisfiable in $(A, a)_{a \in C_0 \cup I}$ let ψ_φ be one of the formulas satisfying (†) for φ. Since $C_0 \cup I$ is countable, there is a countable set C_1 such that $C_0 \subseteq C_1 \subseteq C$ and such that the parameters of ψ_φ are in C_1. Continuing in this way to create sets C_k, let $D = \bigcup\{C_k : k \in \mathbb{N}\}$, and let $q(x)$ be restriction of $p(x)$ to D. As a result, we have:

> (‡) For each $L_{D \cup I}$-formula $\varphi(x)$ that is satisfiable in $(A, a)_{a \in D \cup I}$ there exists $\psi(x) \in q(x)$ such that $\varphi(x) \wedge \neg\psi(x)$ is satisfiable in $(A, a)_{a \in D \cup I}$.

Note that $(a_k : k \in \mathbb{N})$ is also a sequence of indiscernibles in $(A, d)_{d \in D}$, so by the Standard Lemma (Lemma 2.5) there is a model B of T_D that contains a family $(b_\alpha : \alpha < \kappa)$ realising the Ehrenfeucht-Mostowski type of $(a_k : k \in \mathbb{N})$ in $(A, d)_{d \in D}$.

Using Theorem 4.1 we know that there is an L_D-elementary substructure B_1 of B which is atomic over $\{b_\alpha : \alpha < \kappa\}$. We claim that the type $q(x)$ is omitted in B_1. For suppose $q(x)$ is realised in B_1 by some tuple b. We have that $\mathrm{tp}_{B_1}(b/\{b_\alpha : \alpha < \kappa\})$ is isolated so it contains a complete formula $\varphi(x, b_{\alpha_0}, \ldots, b_{\alpha_n})$, where $\varphi(x, y)$ is an L_D-formula and $\alpha_0 < \ldots < \alpha_n < \kappa$. So we have that $\varphi(x, b_{\alpha_0}, \ldots, b_{\alpha_n}) \to \psi(x)$ holds in B_1 for every $\psi(x) \in q(x)$. But since $b_{\alpha_0}, \ldots, b_{\alpha_n}$ and $a_0, \ldots, a_n$ realize the same Ehrenfeucht-Mostowski type over D, we have that

$\varphi(x, a_0, \ldots, a_n) \to \psi(x)$ is valid in B and in A for each formula $\psi(x) \in q(x)$. But that contradicts (‡).

So $q(x)$ is not realised in B_1. Since $|B_1| \geq \kappa$ and L_D is countable, the downward Löwenheim-Skolem Theorem implies that B_1 has an L_D-elementary substructure B_2 of cardinality κ. The type $q(x)$ is also not realised in B_2, so B_2 is a model of cardinality κ which is not κ-saturated (in fact, not even ω_1-saturated). This completes the proof. □

APPENDIX A

Combinatorial principles

A basic combinatorial fact is the *Pigeonhole Principle*:

PROPOSITION A.1. (Pigeonhole Principle) *If an infinite set A is partitioned into finitely many sets $C_1, \ldots, C_k$, then at least one C_i has to be infinite.*

This is quite clear: if each of the C_i would be finite, then A, as the finite union of finite sets, would have to be finite as well. But by repeatedly applying this principle, one can prove statements which are less obvious.

DEFINITION A.2. A partially ordered set $(P, \leq)$ is called a *tree* if P has a least element and for each $p \in P$ the set $p^< = \{q \in P : q < p\}$ is a finite linear order. The size of the set $p^< = \{q \in P : q < p\}$ is the *height* of the element p. The *immediate successors* of p are the elements $r > p$ such that the height$(r) =$ height$(p) + 1$. A tree in which elements $p \in P$ have only finitely many immediate successors is called *finitely branching*. A *branch* of a tree is a maximal linearly ordered subset.

THEOREM A.3. (König's Lemma) *A finitely branching infinite tree has an infinite branch.*

PROOF. If the tree $(P, \leq)$ is infinite, then its least element $p_0 = \perp$ has infinitely many successors. Since the tree is finitely branching, at least one of its immediate successors also has infinitely many successors, by the Pigeonhole Principle. Call this element p_1. By repeating this, we create an infinite sequence of elements $p_0 < p_1 < p_2 < \ldots$ such that p_{i+1} is an immediate successor of p_i and each p_i has infinitely many successors. Then $\{p_0, p_1, p_2, \ldots\}$ is an infinite branch. □

THEOREM A.4. (Ramsey's Theorem) *Let A be infinite and $n \in \mathbb{N}$. Suppose we are given a partition of $[A]^n$, the set of n-element subsets of A, into finitely many subsets $C_1, \ldots, C_k$. Then there is an infinite subset of A all whose n-element subsets belong to the same subset C_i.*

PROOF. We think of the numbers $\{1, \ldots, k\}$ as *colours* and when an n-element subset α of A belongs to C_i, we will say that α has the colour i. And a subset X of A such that all its n-element subsets belong to some fixed C_i will be called *monochromatic*.

We prove Ramsey's Theorem by induction on n. $n = 1$ is just the Pigeonhole Principle. So we assume the statement is true for n and prove it for $n+1$. Let $a_0 \in A$: then any colouring of $[A]^{n+1}$ induces a colouring of $[A \setminus \{a_0\}]^n$: just colour $\alpha \in [A \setminus \{a_0\}]^n$ by the colour of $\{a_0\} \cup \alpha$. We obtain a infinite monochromatic subset $B_1 \subseteq A \setminus \{a_0\}$. Picking an element $a_1 \in B_1$ and continuing in this fashion we obtain an infinitely descending sequence $A = B_0 \supseteq B_1 \supseteq \ldots$ and elements $a_i \in B_i - B_{i+1}$ such that the colour of any $(n+1)$-element subset $\{a_{i(0)}, \ldots, a_{i(n)}\}$ $(i(0) < \ldots < i(n))$ depends only on the value of $i(0)$. Applying the Pigeonhole Principle again there are infinitely many $i(0)$ for which this colour will be the same. These $a_{i(0)}$ then yield the desired monochromatic set. □